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ORIGINAL PAPER

Static equations of the Cosserat continuum derived from intra-granular stresses

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Abstract I present a derivation of the static equations of a granular mechanical interpretation of Cosserat continuum based on a continuum formulated in the intra-granular fields. I assume granular materials with three-dimensional, non-spherical, and deformable grains, and interactions given by traction acting on finite contact areas. Surface traction is decomposed into a mean and a fluctuating part. These account for forces and contact moments. This decomposition leads to a split of the Cauchy stress tensor into two tensors, one of them corresponding to the stress tensor of the Cosserat continuum. Macroscopic variables are obtained by averaging over representative volume. The macroscopic Cauchy stress tensor is shown to be symmetric even in non-equilibrium. The stress tensor of the Cosserat continuum becomes asymmetric when the sum of the contact moments acting on the boundary of the representative volume is different from zero.

Keywords Cauchy stress · Cosserat stress

1 Introduction

One of the greatest mathematicians in history, Emmy Noether, stated what may be one of the most beautiful theorems in physics: "every symmetry gives a conserved quantity". Conservation of energy is linked to symmetry of time; conservation of momentum obeys the translational symmetry of space; and conservation of angular momentum is linked to the isotropy of space, which turns out to lead to the symmetry of the Cauchy stress tensor. Therefore it is not surprising that from the paper of Bardet and Vardoulakis, *On the asymmetry of the stress tensor in granularmatter* [1], so much interest arose among the community of physicists of granular media [2–6].

The question of the symmetry of the Cauchy stress tensor in granular media was intensely debated by Ioannis Vardoulakis, Holger Steeb, Francesco Froiio, and myself while I was in Athens in 2005. The discussion extended further to include one of the Vardoulakis's students, Ioannis Stefanou. Our goal with Ioannis Stefanou was to obtain an explicit expression of the asymmetry of the stress tensor by assuming each grain as a classical continuum. In July 2005 Stefanou sent me a rigorous demonstration of the symmetry of the Cauchy stress tensor that assumed dynamic effects and forces localized at contact points. Later, in April 2007, he sent me new calculations that put in question this symmetry. He showed that the stress tensor in granular media becomes asymmetric when the interaction between grains is decomposed into forces and (non-vanishing) couples.

In a theory of the deformation of continua, an intrinsic surface couple per unit of area is taken into account, in addition to the usual force per unit of area [7]. Two stress tensors are incorporated in this theory: The stress tensor to account for forces; and the couple-stress tensor to account for surface couples. Due to the presence of a couple stress, the stress tensor need not be symmetric. More than one century ago, the Cosserat brothers formulated the kinematics of this "enriched continuum" by assigning both position and orientation to each material point [8]. Their work was the origin ofthe Cosserat theory, which was later extended to a class of generalized continua models, known as the micropolar, microstretch, and micromorphic theories [9,10].

When applying the micropolar theory to granular materials, a major challenge appeared when several authors proposed measures of this stress asymmetry from micromechanical variables: contact forces, contact moments, and branch vectors. Usually, granular materials are regarded as

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discrete systems. Then the continuum is constructed either by time and space averages of discrete quantities [11–14], or by averaging over arbitrary planes that cut through the granular packing [15]. Homogenization strategies can be considered as variants of the method of Irving and Kirkwood [16]. They derived the hydrodynamics equations of particulate systems from classical statistical mechanics, by smearing out particles using a coarse-graining technique. Goddard [17] compared the statistical mechanics method to the classical method of Cauchy for averaging forces in macroscopic surfaces. He proved agreement in both methods, and showed that both involve micropolar effects in strongly inhomogeneous systems.

The homogenization of granular assemblies leads to different expressions, depending on the method of averaging used. A micromechanical definition of stress was given by Christoffersen et al. [18]. They expressed the average of the stress of an assembly of grains in terms of contact forces inside the assembly. Later developments showed that the micromechanical expression for stress may depend on the choice of averaging volume. Material cell partition [19], Dirichlet partition [12], and particle-center-based partition [14] are proposed as the representative volume element (RVE). When quantities are averaged over RVE, the averaged expressions depend on whether the boundary of the volume crosses the contacts between grains, or instead passes through the center of mass of the boundary grains [2]. In a discussion of the paper of Bardet and Vardoulakis on stress asymmetry, Kuhn [3] pointed out that different measures of the stress asymmetry can be obtained simply by shifting the reference points of peripheral grains. Bardet and Vardoulakis acknowledged this remark by replying that that average stress "may be defined non-uniquely" [5].

As regards the microscopic origin of the stress asymmetry, I recall a celebrated discussion during the workshop *From shear bands to rapid flow* in Monte Verita, Switzerland, February 22–27, 2009. In this discussion Vardoulakis pointed out the role of fluctuations in the asymmetry of the stress tensor [20]. Goldhirsch suggested that stress asymmetry stems from the emergence of "localized sources of torques, or angular momentum" [13]. A recent derivation was presented by Luding [21], who concluded that the stress tensor is symmetric in the case of static equilibrium. An early derivation by Goddard [22], showed that in the absence of contact moments the stress tensor remains symmetric, independent of whether grains rotate. These results show that the micromechanical definition of stress, and the origin of the stress asymmetry, are still relevant issues in granular matter.

To derive micromechanical expressions for the stress and couple stress tensors, several authors adopt the concept of an "equivalent continuum" embedded in the "discrete" grains, followed by the use of the virtual work principle [18,23,24]. Bardet and Vardoulakis [5], and Chang and Kuhn [25] agreed that the expression of the stress tensor derived from virtual work is not unique and may lead to different expressions in terms of micromechanics.

The extent of the validity of the continuous models for granular materials when they are derived from point-like particles is still in question: Peters stated that the traditional connection between microscopic particles and macroscopic continuum is not applicable to granular materials because grains themselves are macroscopic [26]. The relevance of the structure of the grains was highlighted by Kruyt [23], who showed that the macroscopic stress tensor obtained from averaging the internal stresses of the grains is different from the one obtained using the concept of equivalent continuum in the discrete system.

In my opinion, precision in the definition of micromechanical quantities can be improved by relaxing the strong assumption that grains are rigid bodies. Instead, micromechanical expressions should be calculated from stresses inside the grains and surface tractions between the grains. Stress and deformation inside the grains can be used to define an equivalent continuum, which may remove ambiguities in the use of the virtual work principle. Details of this continuum are available from either photoelastic experiments [27] or high-resolution numerical models [28], where each grain is regarded as a continuum domain.

In this paper we describe the kinematics and dynamics of individual grains using *intra-granular fields*, which means *fields inside grains*. The aim is to derive an expression for the macroscopic stresses in terms of intra-granular stresses. This connection provides an explanation of the origin of the asymmetry of the stress tensor ingranular media, and allows us to calculate the stress asymmetry in terms of contact moments.

2 Notation and definitions

The following notation will be used: The elements of a 3×3 matrix $\mathbf{A}$ are represented by A_{ij}; the elements of a 3D vector $\mathbf{x}$ are x_j. The Einstein summation convention is used for expressions involving indices, which means that repeated indices represent a sum over all possible values for that index. The Levi-Civita permutation tensor is used: e_{ijk} is 1 when the indices are in cyclic order; -1 when they are in anti-cyclic order; and 0 otherwise. For example, if $\mathbf{a} = \mathbf{b} \times \mathbf{c}$ is the vectorial product in 3D, the indices of $\mathbf{a}$ satisfy $a_i = e_{ijk} b_j c_k$. The time derivative (material derivative) of a is denoted by $da/dt = \dot{a}$ and its partial derivative with respect to x_i is denoted by $\partial a / \partial x_i = a_{,i}$.

We will assume the existence of a continuum at two different length scales as show in Fig. 1. The *macroscopic continuum* in Fig. 1a describes the system at the macroscale, by averaging the intra-granular field over RVEs V' shown in Fig. 1b. The *intra-granular continuum* is given by the stress and deformation inside the grains. This continuum will

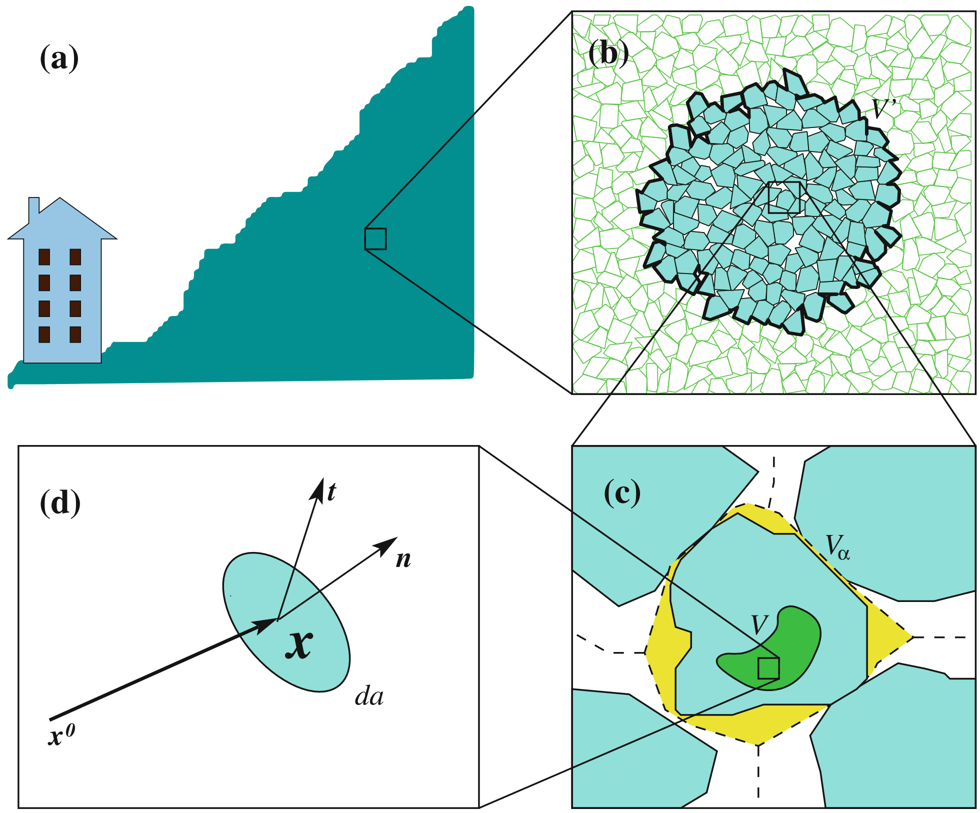

Fig. 1 Continuum fields in a macroscopic domain as shown in (**a**) are derived by averaging intra-granular quantities over representative volume (RVE) as shown in (**b**). The boundary of V' passes through the contact surfaces of the peripheral grains of this volume. The RVE is tessellated in sub-domains. In (**c**) the yellow region is the subdomain V_α containing the α-grain. The *green* region in (**c**) shows an intra-granular volume $V \subseteq V_\alpha$. A point inside V is shown in (**d**).The force $d\mathbf{F}$, acting on the area da oriented in the direction of the unit vector $\mathbf{n}$ is used to define the traction at $\mathbf{x}$ as $\mathbf{t}(\mathbf{x}, \mathbf{t}) = d\mathbf{F}/da$. The area is centered on the "test point" $\mathbf{x}$. The vector $\mathbf{x}^0$ is called "reference point"

be assumed to be classical. This means that the kinematics of material points inside grains is given by translation only, and that the interaction in every imaginary surface inside the grains is given by a continuous distribution of traction, which is force per unit of area. Variation of intra-granular stresses takes place at length scales much smaller that the diameter d of the grains. Details of the intra-granular field is not required in the investigation of the macroscopic behaviour. To make the averaging process meaningful, we need to choose RVEs whose diameter D satisfies $d \ll D < \Lambda$, where Λ is the characteristic macroscopic length scale.

We will assume that the intra-granular stress is continuous inside each grain and vanishes outside grains. Discontinuities in the stress may exist in each interface between two grains in contact. To avoid these discontinuities in our continuous description, we assume that the RVE does not pass through the inner point of the grains. Instead the boundary passes through the contact areas of the peripheral grains. We adopt also the material cell partition proposed by Bagi [29]. This partition is a tessellation of the RVE in sub-domains, where each domain contains exactly one grain. Stress is assumed to be continuous inside each sub-domain. The use of the divergence theorem in the subdomains will allow us to obtain micromechanical expressions forthe stress asymmetry in terms of quantities defined along the contact areas, such as contact forces and contact moments.

3 Static equations of intra-granular continuum

In this section we define the equations of the intra-granular continuum in a sub-domain $V_\alpha \subset V'$ of the partition of V'. First we define the kinematic quantities of this continuum. We use the Eulerian formulation for the flow field, so that the intra-granular fields are depicted as a function of fixed position $\mathbf{x}$ and time t. For each *test point* $\mathbf{x} \in V_\alpha$ there is a velocity $\mathbf{v}(\mathbf{x}, \mathbf{t})$ and a mass density $\rho(\mathbf{x}, \mathbf{t})$. Let us consider an infinitesimal-element volume dV containing $\mathbf{x}$. The linear momentum of this volume is $d\mathbf{P} = \rho \mathbf{v} dV$ and its angular momentum with respect to a *reference point* $\mathbf{x}^0$ is $d\mathbf{L} = \rho \boldsymbol{\ell} \times \mathbf{v} dV$, where $\boldsymbol{\ell} = \mathbf{x} - \mathbf{x}^0$.

Now we define the dynamic quantities of the continuum. Let us consider an infinitesimal area da centered on the test point $\mathbf{x}$ and oriented in the direction $\mathbf{n}$ as shown in Fig. 1d. Let us define $d\mathbf{F}$ as the surface force acting on this area. The traction is defined as the stress-vector, given by the force per unit of area $\mathbf{t}(\mathbf{x}, \mathbf{t}) = d\mathbf{F}/da$. Since we assumed a classical continuum at the intra-granular level, there is no intrinsic couple per unit of area. The Cauchy stress tensor is related to the traction by

$$n_k \sigma_{kj} = t_j. \tag{1}$$

Now we set out the conservation equations on an arbitrary volume $V \subseteq V_\alpha$. We assume that all fields are continuously differentiable inside V_α. We denote the boundary surface by ∂V_α. We call M the mass contained in the volume; $\mathbf{F}$ and $\mathbf{T}$ the force and torque acting on V; $\mathbf{P}$ and $\mathbf{L}$ the linear and angular momentum of V. The body force acting on dV is given by $\rho\mathbf{g}(\mathbf{x})dV$. The balance equation of mass $\dot{M} = 0$ reads

$$\frac{d}{dt}\int_V \rho dV = 0, \quad (2)$$

the balance equation of linear momentum $\dot{\mathbf{P}} = \mathbf{F}$ accounting for both surface and body forces reads

$$\frac{d}{dt}\int_V \rho v_j dV = \int_V \rho g_j dV + \int_{\partial V} t_j da, \quad (3)$$

and the balance of angular momentum $\dot{\mathbf{L}} = \mathbf{T}$ reads

$$\frac{d}{dt}\int_V \rho e_{jkl}\ell_k v_l dV = \int_V \rho e_{jkl}\ell_k g_l dV + \int_{\partial V} e_{jkl}\ell_k t_l da. \quad (4)$$

Taking into account that the equations above are valid for any $V \subseteq V_\alpha$ we will derive the differential form of these balance equations: from Eq. (2) and the Reynolds transport theorem [30] we obtain the continuity equation $\dot{\rho} + \rho v_{k,k} = 0$. Applying the Reynolds transport theorem to the time derivatives in Eqs. (3) and (4), and using the continuity equation we get

$$\int_V \rho(g_j - \dot{v}_j)dV + \int_{\partial V} t_j da = 0, \quad (5)$$

$$\int_V \rho e_{jkl}\ell_k(g_l - \dot{v}_l)dV + \int_{\partial V} e_{jkl}\ell_k t_l da = 0. \quad (6)$$

These balance equations can be used to derive the differential equation for the Cauchy stress tensor [30]. The equation for linear momentum balance reads

$$\sigma_{ij,i} + \rho(g_j - \dot{v}_j) = 0, \quad \mathbf{x} \in V_\alpha, \quad (7)$$

with boundary condition

$$n_k\sigma_{kj} = t_j, \quad \mathbf{x} \in \partial V_\alpha. \quad (8)$$

The equation for angular momentum balance results in

$$e_{jkl}\sigma_{kl} = 0, \quad \mathbf{x} \in V_\alpha. \quad (9)$$

Which is equivalent to $\sigma_{jk} = \sigma_{kj}$. This proves that the symmetry of the intra-granular Cauchy stress tensor is a direct consequence of the conservation of angular momentum. This symmetry is independent of the constitutive model, the nature of the surface traction, and whether the system is in static equilibrium.

We note that any asymmetry of the Cauchy stress tensor will imply a violation of the conservation of angular momentum. Yet the formulation of the Cosserat continuum leads inexorably to a stress tensor that is asymmetric. This apparent contradiction will be solved by establishing a relation between the Cauchy stress tensor and the stress tensor in the Cosserat continuum.

4 Decomposition of traction

In this section we will derive a special form of micropolar continuum by splitting the traction in two components, accounting forces and contact moments (couples). We recall now the definition of the Cauchy stress tensor in Eq. (1) that is given in terms of the total traction. The traction at the contact area a_c is used to define a contact force and a contact moment (couple) as

$$F_j^c = \int_{a_c} t_j da, \quad (10)$$

$$\kappa_j^c = \int_{a_c} e_{jkl}(x_k - x_k^c)t_l da, \quad (11)$$

where $\mathbf{x}^c$ is the centroid of the contact area. Equation (10) accounts for normal and shear force, while Eq. (11) is a contact moment that accounts for a rolling (bending) and a twisting moment. Following this natural decomposition of contact interaction into a force and a couple, we split the traction acting at each contact into two components as shown in Fig. 2:

$$t_j^c(\mathbf{x}) = t_j^{cm}(\mathbf{x}) + t_j^{cf}(\mathbf{x}), \quad \mathbf{x} \in a_c \quad (12)$$

$$t_j^{cm}(\mathbf{x}) = f^c(\mathbf{x})F_j^c, \quad (13)$$

$$t_j^{cf}(\mathbf{x}) = t_j^c(\mathbf{x}) - f^c(\mathbf{x})F_j^c, \quad (14)$$

where $f(\mathbf{x})$ is a test function that is continuously differentiable in a_c. This function vanishes in the boundary of a^c and satisfies

$$\int_{a_c} f^c(\mathbf{x})da = 1, \quad \int_{a_c} x_j f^c(\mathbf{x})da = x_j^c. \quad (15)$$

We will call $\mathbf{t}^{cm}$ and $\mathbf{t}^{cf}$ *mean* and *fluctuating parts* of the tractions at a_c. Using Eqs. (14) and (15) we can show that the mean part produces a force and no couple. The fluctuating part has vanishing integral

$$\int_{a_c} t_j^{cf}(\mathbf{x})da = 0, \quad (16)$$

so that it does not produce forces, yet it accounts for the couple by

$$\kappa_j^c = \int_{a_c} e_{jkl}(x_k - x_k^c)t_l^{cf}(\mathbf{x})da = \int_{a_c} m_j^c(\mathbf{x})da. \quad (17)$$

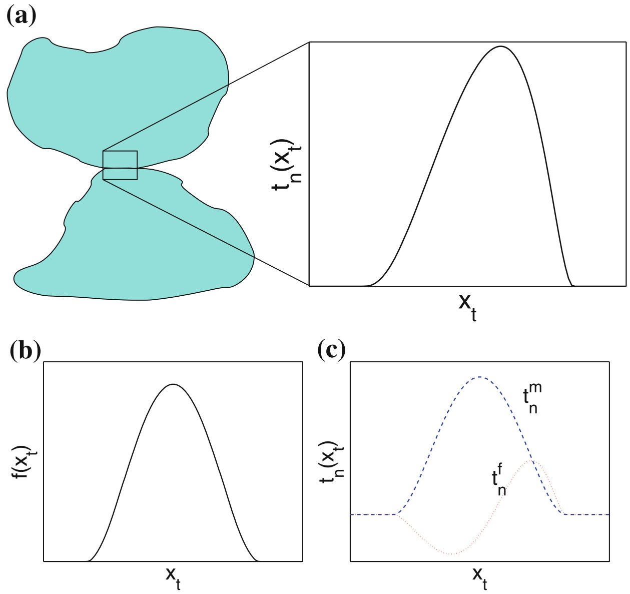

Fig. 2 Schematic representation of **a** the normal component of traction along the contact area, **b** test function along the tangential direction of the contact area, and **c** mean and fluctuating components of this traction

Here we define the *couple distribution* along the contact area a_c as

$$m_j^c(\mathbf{x}) = e_{jkl}(x_k - x_k^c)t_l^{cf}(\mathbf{x}). \tag{18}$$

Note that this distribution is independent of the referent point $\mathbf{x}^0$. Now we extend this decomposition of traction to every point on ∂V_α as follows:

$$t_j(\mathbf{x}) = t_j^m(\mathbf{x}) + t_j^f(\mathbf{x}), \qquad \int_{\partial V_\alpha} t_j^f(\mathbf{x})da = 0, \tag{19}$$

where

$$t_j^m(\mathbf{x}) = \begin{cases} t_j^{cm}(\mathbf{x}) & \text{if } \mathbf{x} \in a_c \subset \partial V_\alpha \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

and $\mathbf{t}^f = \mathbf{t} - \mathbf{t}^m$. This decomposition leads to a split of the intra-granular Cauchy stress tensor,

$$\begin{aligned} &\sigma_{ij}(\mathbf{x}) = \sigma_{ij}^m(\mathbf{x}) + \sigma_{ij}^f(\mathbf{x}), \qquad \mathbf{x} \in \partial V_\alpha, \\ &n_i\sigma_{ij}^m(\mathbf{x}) = t_j^m(\mathbf{x}), \quad n_i\sigma_{ij}^f(\mathbf{x}) = t_j^f(\mathbf{x}). \end{aligned} \tag{21}$$

This decomposition defines two independent stress tensors. The first one is related to the forces between the grains, and the second one accounts for contact moments between grains. An interesting connection of this decomposition with micro-polar theory arises when we use $\sigma^{\mathbf{f}}$ to define a new tensor:

$$\mu_{ij}(\mathbf{x}) = \begin{cases} e_{jkl}(x_k - x_k^c)\sigma_{kl}^f & \text{if } \mathbf{x} \in a_c \subset \partial V_\alpha \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Using Eq. (22) it is easy to prove that this tensor satisfies a property similar to the couple stress tensor

$$n_k\mu_{kj}(\mathbf{x}) = m_j(\mathbf{x}) \quad \mathbf{x} \in \partial V_\alpha, \tag{23}$$

when the couple distribution along ∂V_α is defined as

$$m_j(\mathbf{x}) = \begin{cases} m_j^c(\mathbf{x}) & \text{if } \mathbf{x} \in a_c \subset \partial V_\alpha \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Now we seek the partial differential equations for the tensors defined above. Let us assume that $\sigma^{\mathbf{m}}$, $\sigma^{\mathbf{f}}$, and μ are continuously differentiable in V_α. Then the equations for these tensors are derived using the balance equations. First we develop the surface integral of the traction. Using Eqs. (19) and (21) we get

$$\int_{\partial V_\alpha} t_j da = \int_{\partial V_\alpha} t_j^m da = \int_{\partial V_\alpha} n_i\sigma_{ij}^m da = \int_{V_\alpha} \sigma_{ij,i}^m dV. \tag{25}$$

Substituting this equation into Eq. (5) for $V = V_\alpha$ we get the momentum balance equation for $\sigma^{\mathbf{m}}$:

$$\int_{V_\alpha} \left[\sigma_{ij,i}^m + \rho(g_j - \dot{v}_j)\right] dV = 0. \tag{26}$$

This is the momentum balance equation for V_α. A sufficient condition for satisfying Eq. (26) is

$$\sigma_{ij,i}^m + \rho(g_j - \dot{v}_j) = 0, \quad \mathbf{x} \in V_\alpha. \tag{27}$$

Assuming that the Cauchy stress is split as $\sigma = \sigma^{\mathbf{m}} + \sigma^{\mathbf{f}}$ inside V_α and using Eqs. (7) and (27), we obtain the differential equation for $\sigma^{\mathbf{f}}$:

$$\sigma_{ij,i}^f = 0 \quad \mathbf{x} \in V_\alpha. \tag{28}$$

Now we will derive the angular momentum equation for V_α. First we split the surface integral in Eq. (6) for $V = V_\alpha$

$$\int_{\partial V_\alpha} e_{jkl}\ell_k t_l da = \int_{\partial V_\alpha} e_{jkl}\ell_k t_l^m da + \int_{\partial V_\alpha} e_{jkl}\ell_k t_l^f da, \tag{29}$$

and then developing each term of this equation:

$$\begin{aligned} \int_{\partial V_\alpha} e_{jkl}\ell_k t_l^m da &= \int_{\partial V_\alpha} e_{jkl}\ell_k n_i \sigma_{il}^m da \\ &= \int_{V_\alpha} (e_{jkl}\ell_k\sigma_{il}^m)_{,i} dV \\ &= \int_{V_\alpha} e_{jkl}(\sigma_{kl}^m + \ell_k\sigma_{il,i}^m)dV, \end{aligned} \tag{30}$$

$$\begin{aligned} \int_{\partial V_\alpha} e_{jkl}\ell_k t_l^f da &= \int_{\partial V_\alpha} m_j da = \int_{\partial V_\alpha} n_i\mu_{ji} da \\ &= \int_{V_\alpha} \mu_{ij,i} dV, \end{aligned} \tag{31}$$

and substituting these two equations into Eq. (29) we get

$$\int_{\partial V_\alpha} e_{jkl}\ell_k t_l da = \int_{V_\alpha} (\mu_{ij,i} + e_{jkl}\sigma^m_{kl} + e_{jkl}\ell_k\sigma^m_{il,i})dV. \quad (32)$$

By substituting this equation into Eq. (6) and using Eq. (26) we obtain the equation for angular momentum of V_α

$$\int_{V_\alpha} \left[\mu_{ij,i} + e_{jkl}\sigma^m_{kl}\right] dV = 0. \quad (33)$$

A sufficient condition to satisfy Eq. (33) is

$$\mu_{ij,i} + e_{jkl}\sigma^m_{kl} = 0 \quad \mathbf{x} \in V_\alpha. \quad (34)$$

Equations (27) and (34) correspond to a special case of micropolar continuum. In the absence of body forces, and in equilibrium, these equations reduce to

$$\sigma^m_{ij,i} = 0, \quad \mu_{ij,i} + e_{jkl}\sigma^m_{kl} = 0, \quad (35)$$

which correspond to the balance equations of the Cosserat continuum [20]. This correspondence requires us to assign $\sigma^{\mathbf{m}}$ as the stress tensor of the Cosserat continuum —called here Cosserat stress—and μ as the couple stress tensor.

We also note that in the absence of fluctuations in traction the Cosserat stress tensor coincides with the Cauchy stress tensor, and it is therefore symmetric. Fluctuations of the traction at the contact surfaces may lead to $\sigma^{\mathbf{f}} \neq \mathbf{0}$ and hence to a stress tensor that differs from the Cauchy stress tensor. The asymmetry of the Cosserat stress tensor is derived from Eq. (9):

$$e_{ijk}\sigma^m_{jk} = -e_{ijk}\sigma^f_{jk}. \quad (36)$$

Therefore $\sigma^m_{ij} - \sigma^m_{ji}$ does not necessarily vanish, which is consistent with the Cosserat theory.

5 Macroscopic equations

In this section we will formulate the static equations of the macroscopic continuum by averaging the intra-granular stress over RVE. The aim is to derive two different representations of the same system. One of them is obtained by averaging the static equations of the Cauchy stress tensor. The other is obtained by averaging the static equations of the Cosserat stress tensor. We will also prove that both representations reduce to the Boltzmann (classical) continuum when couples are absent.

We assume a partition of the RVE V' is given by the collection of sets $\{V_\alpha\}$ [19,29]. Then the average $\bar{a}$ of any quantity a over V' is defined as

$$\bar{a} = \frac{1}{V'}\sum_{V_\alpha \subset V'} V_\alpha \bar{a}^\alpha, \quad \bar{a}^\alpha = \frac{1}{V_\alpha}\int_{V_\alpha} a dV. \quad (37)$$

The static equations of the macroscopic Cauchy stress tensor are derived by averaging Eqs. (7) and (9) over V':

$$\begin{aligned} &\bar{\sigma}_{ij,i} + \rho'(g'_j - a'_j) = 0, \\ &\bar{\sigma}_{ij} = \bar{\sigma}_{ji}, \end{aligned} \quad (38)$$

where $\rho' = \bar{\rho}$ is the macroscopic density, $\mathbf{a}' = \overline{\rho\dot{\mathbf{v}}}/\rho'$ is the macroscopic acceleration, $\mathbf{g}' = \overline{\rho\mathbf{g}}/\rho'$ the macroscopic body force. Equation (38) corresponds to the static equations of the *total stress*, which is the most important in continuum mechanics, as it accounts for both forces and couples in the intergranular interaction. We note that from the symmetry of the intra-granular Cauchy stress tensor in Eq. (9), the total stress is symmetric too. This symmetry is independent of dynamic effects or existence of body forces. This result is consistent with early derivations of Bagi [19], Kruyt [23], and Luding [21], who showed that the averaged stress tensor is symmetric under more restricted assumptions.

An equivalent formulation of the static equations of the macroscopic continuum in terms of $\sigma^{\mathbf{m}}$ and μ can be retrieved by using Eqs. (26) and (33) and the average operator defined by Eq. (37). We obtain

$$\begin{aligned} &\overline{\sigma^m}_{ij,i} + \rho'(g'_j - a') = 0 \\ &\overline{\mu}_{ij,i} + e_{jkl}\overline{\sigma^m}_{kl} = 0. \end{aligned} \quad (39)$$

These equations correspond to the static equations of the Cosserat continuum. Thus we have proven that the static equations of the total stress are equivalent to the static equation of the Cosserat continuum. The only difference between the two formulations is that the latter one makes an explicit decomposition of stress into two components, accounting for forces and couples.

6 Asymmetry of the stress tensor in granular media

We will calculate the stress asymmetry by decomposing the total stress into surface and volumetric parts. This decomposition has been proposed by Bagi [19] and Luding [21] for the Cauchy stress tensor. Here we decompose also the Cosserat stress tensor, to obtain an explicit expression for the asymmetry of the macroscopic Cosserat stress tensor $\overline{\sigma^{\mathbf{m}}}$. We use the identity $\sigma_{ij} = (\ell_i\sigma_{kj})_{,k} - \ell_i\sigma_{kj,k}$ and the divergence theorem, and the averaged stress on $V = V_\alpha$ given by Eq. (37) for $V = V_\alpha$ results:

$$\bar{\sigma}^\alpha_{ij} = \frac{1}{V_\alpha}\int_{\partial V_\alpha} \ell_i n_k \sigma_{kj} da + \frac{1}{V_\alpha}\int_{V_\alpha} \ell_i \sigma_{kj,k} dV. \quad (40)$$

Using this identity, Eqs. (7) and (1) we get

$$\bar{\sigma}^\alpha_{ij} = \frac{1}{V_\alpha}\int_{\partial V_\alpha} \ell_i t_j da + \frac{1}{V_\alpha}\int_{V_\alpha} \rho\ell_i(\dot{v}_j - g_j)dV. \quad (41)$$

Following a similar procedure, (26) and (21) lead to the macroscopic Cosserat stress tensor:

$$\overline{\sigma^m}^{\alpha}_{ij} = \frac{1}{V_\alpha}\int\limits_{\partial V_\alpha} \ell_i t^m_j da + \frac{1}{V_\alpha}\int\limits_{V_\alpha} \rho \ell_i (\dot{v}_j - g_j) dV. \quad (42)$$

Subtracting the last two equations results in

$$\overline{\sigma^f}^{\alpha}_{ij} = \frac{1}{V_\alpha}\int\limits_{\partial V_\alpha} \ell_i t^f_j da. \quad (43)$$

Now we can use Eq. (36) to obtain a measure of the asymmetry of the macroscopic Cosserat stress tensor:

$$e_{ijk}\overline{\sigma^m}^{\alpha}_{jk} = -e_{ijk}\overline{\sigma^f}^{\alpha}_{jk} = -\frac{1}{V_\alpha}\int\limits_{\partial V_\alpha} e_{ijk}\ell_j t^f_k da. \quad (44)$$

Thus the asymmetry arises from the fluctuating part of the traction. Since the surface ∂V_α goes around the grain, through its contact areas, we can use of Eq. (17) to relate the asymmetry to the contact moments,

$$e_{ijk}\overline{\sigma^m}^{\alpha}_{jk} = -\frac{1}{V_\alpha}\sum_{a_c \subset \partial V\alpha} \kappa^c_i. \quad (45)$$

We plug this result into Eq. (37) and use the fact that $\mathbf{k}^{\alpha\beta} + \mathbf{k}^{\alpha\beta} = 0$, where $\mathbf{k}^{\alpha\beta}$ is the couple acting on grain α due to grain β, we obtain

$$e_{ijk}\overline{\sigma^m}_{jk} = -\frac{1}{V'}\sum_{a_c \subset \partial V'} \kappa^c_i. \quad (46)$$

In other words, the Cosserat stress tensor is asymmetric when the sum of the contact moments acting on the boundary of the RVE is different from zero. Note that the asymmetry holds for static conditions, while it disappears when interaction is given by contact forces only. The asymmetry may hold also in the case where the interaction given by traction is replaced by punctual forces and localized contact moments.

To compare this result with previous work, we express the time derivative of the angular momentum in the RVE in terms of body forces, and forces and couples acting on its boundary

$$\dot{L}^{V'}_i = C^{V'}_i + \sum_{a_c \subset \partial V'} \left(\kappa^c_i + e_{ijk}\ell^c_j F^c_k\right), \quad (47)$$

where $\mathbf{L}^{V'}$ and $\mathbf{C}^{V'}$ are the angular momentum and the torque produced by the body forces on the RVE. Replacing Eq. (47) into Eq. (46) results:

$$e_{ijk}\overline{\sigma^m}_{jk} = \frac{1}{V'}\left(C^{V'}_i - \dot{L}^{V'}_i + \sum_{a_c \subset \partial V'} e_{ijk}\ell^c_j F^c_k\right). \quad (48)$$

In the case of static equilibrium, this result is consistent with the derivation of Kruyton the asymmetry of the "averaged homogenized stress tensor" [23]. If we neglect dynamics and body forces, we obtain the same expression for the stress asymmetry given by other authors [6,12,25,31].

A direct conclusion of Eq. (46) is that the stress tensor in the Cosserat continuum is symmetric when there are no contact moments acting on the grains. In the literature, the macroscopic equations for such a system is defined as *Boltzmann continuum* [6,20]. In this continuum, grains interact via contact forces only. In this case $\sigma^{\mathbf{f}} = \mathbf{0}$, so that the total stress coincides with the Cosserat stress tensor. Thus both Eqs. (38) and (39) reduce to the same set of equations:

$$\overline{\sigma^m}_{ij,i} + \rho'(g'_j - a'_j) = 0,$$
$$\overline{\sigma^m}_{ij} = \overline{\sigma^m}_{ji}. \quad (49)$$

These are the static equations of the Boltzmann continuum. Note that they are not the same as the static equations of the total stress. This tensor accounts the total traction that involves forces and couples calculated via Eqs. (10) and (11). On the other hand, the stress tensor of the Boltzmann continuum assumes that there is no fluctuating component in the traction, which implies that there is no couples. This result is consistent with early formulation of Vardoulakis and Sulem [20], that shows that the Boltzmann continuum is the special case of Cosserat continuum when there are no couples.

7 Conclusions

The balance equations for stress and the couple stress tensors have been derived by averaging intra-granular stresses over representative volume. Surface traction at contact areas was decomposed into mean and fluctuating parts. This led to a decomposition of the Cauchy stress tensor into two tensors, accounting for intergranular forces and contact moments. While the Cauchy stress tensor is always symmetric, the stress tensor of the Cosserat continuum can be asymmetric. In particular, the macroscopic version of this tensor becomes asymmetric when the sum of contact moments on the boundary of the RVE is different from zero.

We have proven equivalence between the equations of the total stress and the static equations of the Cosserat theory when interpreted for granular systems. While the equations for the total stress arise from direct homogenization of the intra-granular stresses, the static equations of the Cosserat theory are derived from the explicit decomposition of the Cauchy stresss tensor into one tensor accounting for forces and the other accounting for couples.

The derivation of the static equations of the Cosserat continuum using intra-granular fields presents an alternative view to the work of Goldhirsch [13], who used a "coarse-graining" function to average discrete quantities.Goldhirsch assumed that the interaction is given by discrete contact forces, while we assume interaction given by continuous tractions. An important step in future research is to find a relation between the "coarse-graining" approach and the "intra-granular" averaging approach.

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